The Mathematics of Musical Scales

In this activity, students will explore the mathematical principles of musical scales by measuring and graphing the frequencies produced by standard-pitched instruments. Students will use technology to graph and determine the mathematical formula which controls the musical scale. With modifications, this activity is suitable for middle-school or high school students. Fundamentally, one of the most musically pleasing intervals (excluding unison or 1:1) is the perfect octave, where upper note has exactly twice the frequency of the lower note, giving a ratio of 1:2. The next most pleasing interval is the perfect 5th, with a ratio of 1:1.5. Many consider the major 3rd to be next, with a ratio of 1:1.33. Indeed these three intervals (tonic, major 3rd, perfect 5th) make up the major triad, the basis for all musical chords. A pattern emerges. If 12 musical half-tones are needed to fill in an octave, hitting on or very near the above ratios for the major 3rd and perfect 5th, it soon becomes apparent that the intervening frequencies cannot be simply divided by 12. The musical scale is non-linear. In fact, it is built upon the 12th root of 2, as there are twelve half-tones needed to double a frequency to reach its octave above. This ratio is very close to that of 18/17, which is considered close enough for the layout of frets on a stringed instrument. This non-linearity is apparent when inspecting the shape of a piano's harp.

Learning Objectives:

1. Measure and plot the frequencies of different musical notes.
2. Inspect a graph to determine if the musical scale is linear or non-linear.
3. Use a graphing calculator to determine the formula for the musical scale.

Standards:

CCSS.Math.Content.HSF-IF.C.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

CCSS.Math.Content.HSF-LE.A.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

Materials Required:

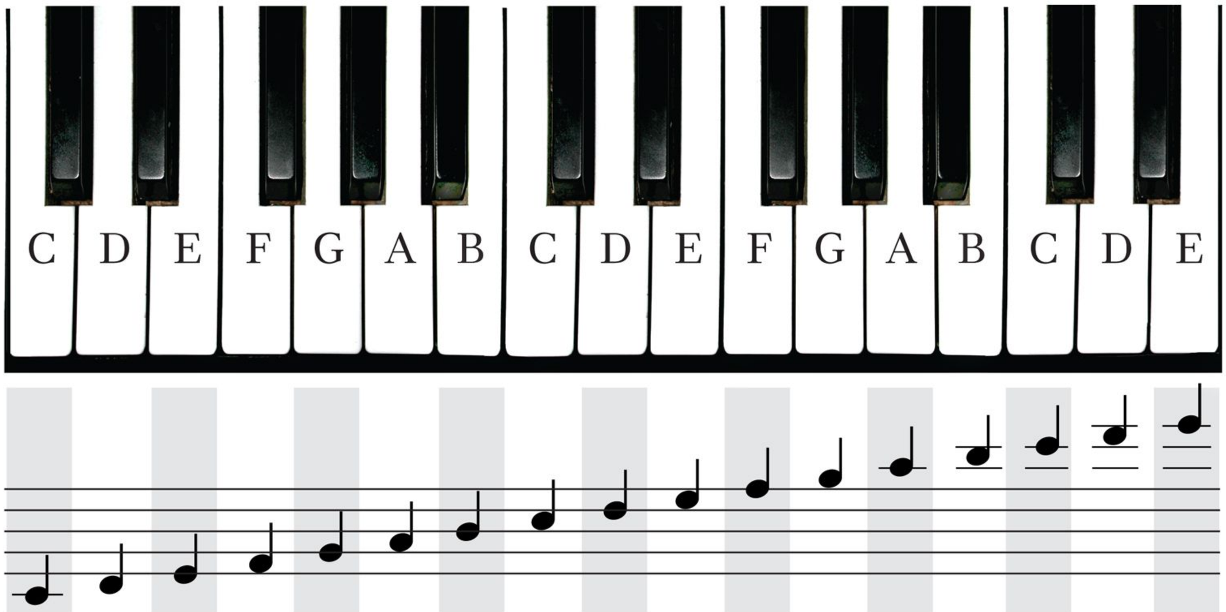
1. Smartphone, tablet or comparable device with frequency counting application
2. Electric piano or digital keyboard, and/or a guitar
3. Graphing calculators
4. MS Excel or a comparable spreadsheet application

Safety:

Basic eye protection.

References:

Physics of Music - Notes: http://pages.mtu.edu/~suits/notefreqs.html
Graphic of Keyboard Layout and Frequencies:
https://c1.staticflickr.com/9/8011/7529666092_f8e7311a68_b.jpg
Origins of Western Musical Scales:
https://golem.ph.utexas.edu/category/2010/02/a_look_at_the_mathematical_ori.html



Activity:

After learning the concept of frequency as cycles/second and its unit (Hertz), the students should explore the nature of musical scales by playing C4 on a piano and octaves (C5 and C3) above and below. The students will note the similarity in pitch between any note and the perfect octaves which bracket it. The students will count and record the number of keys (12), both black and white, between any note and its corresponding octave both above and below. Corresponding discovery can be achieved on a guitar, noting the number of frets (12) between octaves, and measuring the length of an open string and its fretted octave above, yielding a ratio of 1:2.

Once the students have had time to play with scales and intervals, have them derive a hypothesis: Each succeeding note **(is/is not)** higher than its lower neighbor note by a constant ratio. Or: Each note **(is/is not)** a specific number of hertz higher than its lower neighbor. Students can derive their own hypotheses.

Next, collect several keyboards (and guitars!) and arrange them in groups around the room.

After a suitable period of discovery, set up a frequency-counting application at each station. Have the students play the C4 note into the frequency-counting application, observe the waveform and record the frequency. The students should then repeat this process with every key (black and white) up to the C5 key. Have the students repeat this process with E5, G5 and C6.

The students should use Excel, or a comparable spreadsheet application, to create a table with the following as column headings. (Key #, Note, Frequency, Change in frequency, Ratio to Previous note, and Ratio to C4). See example below.

Once the students have done this with the piano, have them do the same with the guitar and create a table as well. They should correlate the pitch of the open guitar strings to their corresponding piano keys.

Modification: If no musical instruments are available, a concert pitch note-frequency table can give the students a less “hands-on” experience.

Key #	note	frequency	Change in frequency	Ratio to previous note	Ratio to C4

With this data, students can embark on many graphing adventures. Ideally students will discover that the relationships are non-linear, and may even notice the function is logarithmic. Finally, students can plug values from the table into a graphing calculator which should calculate the function as y= $\sqrt[12]{2x}$.

Name ________________________________

Assessment
The Mathematics of Musical Scales

1. The graph of musical note frequencies is…

Linear -or- Non-linear

2. How many white keys are between any white note on the piano and the octave above (not counting the octave key)?

A. 5
B. 7
C. 12
D. 16

3. How many black keys are between any black note and it's octave above (not counting the octave key)?

A. 5
B. 7
C. 12
D. 16

4. How many total keys are there between any key and its octave above (not counting the octave key)?

A. 5
B. 7
C. 12
D. 16

5. The note A4 has a frequency of 440 Hz. A5, one octave above, has a frequency of…

A. 220 Hz
B. 880 Hz
C. 660 Hz
D. 1320 Hz

6. The note G4 has a frequency of 392 Hz. Calculate the frequency of D5, a perfect fifth above G4.

A. 588 Hz
B. 784 Hz
C. 392 Hz
D. 261 Hz

7. The note C7 has a frequency of 2093 Hz. Calculate the frequency of F7, a major third above C7.

A. 2794 Hz
B. 3136 Hz
C. 3520 Hz
D. 7040 Hz

8. Each note is lower than the subsequent note by a constant ratio.

True -or- False

9. The ratio between the frequency of a note and that of one octave higher is…

A. 1:1.25
B. 1:1.33
C. 1:1.5
D. 1:2

10. The ratio between the frequency of a note and that of one a perfect fifth higher is…

A. 1:1.25
B. 1:1.33
C. 1:1.5
D. 1:2

Assessment Key:

1. Non-linear
2. B - 7
3. A - 5
4. C - 12
5. B - 880 Hz
6. A - 588 Hz
7. A - 2794 Hz
8. False
9. D - 1:2
10. C - 1:1.5

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